Canonical and D-transformations in Theories with Constraints

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Abstract

We describe a class of transformations in a super phase space (we call them D-transformations), which play in theories with second-class constraints the role of ordinary canonical transformations in theories without constraints. Namely, in such theories they preserve the forminvariance of equations of motion, their quantum analogue are unitary transformations, and the measure of integration in the corresponding hamiltonian path integral is invariant under these transformations.

I. INTRODUCTION

As is well known canonical transformations play an important role in the hamiltonian formulation of classical mechanics without constraints [1]. They preserve the forminvariance of the hamiltonian equations of motion and their quantum analogue are unitary transformations [2,3]. Canonical transformations constitute also a powerful tool of the classical mechanics, which allows one often to simplify solution of the theory. For example, it is enough to mention that evolution is also a canonical transformation. Quantum implementation of canonical transformations where discussed in numerous papers, see for example [4–7]. However, modern physical theories in their classical versions are mostly singular (in particular, gauge) ones, which means that in the hamiltonian formulation they are theories with constraints [8,9]. Equations of a hamiltonian theory with constraints are not form invariant under canonical transformations, but namely this circumstance allows one to use these transformations to simplify the equations and to clarify the structure of the gauge theory in hamiltonian formulation. Moreover, formulations of a gauge theory in two diffrent gauges are connected by means of a canonical transformation [9,10]. In general case, equations of constraints change their form ander the canonical transformations. That is an indirect indication that the quantum version of the canonical transformations in constrained theories is not an unitari transformation (Of course, we are speaking about the complete theory, but not about its reduced unconstrained version). Thus, in case of constrained theories one can believe that besides of the canonical transformation another kind of transformations has to exist, which preserves the form invarians of the equations of motion and which induces unitary transformations on the quantum level. Namely they play the role of ordinary canonical transformations in theories without constraints.

In this paper we describe such kind of transformations for theories with second-class constraints, which is, in fact, a general case, because of a theory with first-class constraints can be reduced to a theory with second-class ones by a gauge fixing. We call such transformation D-transformations.

II. GENERALIZED CANONICAL TRANSFORMATIONS

Let a classical mechanics be given with phase variables $\eta = (\eta^A), \quad A = 1, \ldots, 2n$ (in general case they belong to Berezin algebra [11,9] and have the Grassmann parities $P(\eta^A) = P_A$), and with a symplectic metrics $\Lambda^{AB}(\eta)$, which defines a generalized super Poisson bracket for any two functions $F(\eta)$ and $G(\eta)$ with definite Grassmann parities $P(F)$ and $P(G)$,

$$\{F, G\}^{(\eta,\Lambda)} = \frac{\partial_r F}{\partial \eta^A} \Lambda^{AB}(\eta) \frac{\partial_l G}{\partial \eta^B} , \tag{2.1}$$

where $\partial_r/\partial\eta^A$ and $\partial_l/\partial\eta^B$ are the right and left derivatives respectively. The metrics $\Lambda^{AB}(\eta)$ is a T_2-antisymmetric supermatrix [9], $P(\Lambda^{AB}) = P_A + P_B, \quad \Lambda^{AB}(\eta) = -(-1)^{P_A P_B} \Lambda^{BA}(\eta)$, obeying the conditions,

$$(-1)^{P(A)P(C)} \Lambda^{AD}(\eta) \frac{\partial_l \Lambda^{BC}(\eta)}{\partial \eta^D} + \text{cycl.}(A, B, C) = 0 , \tag{2.2}$$

which are necessary and sufficient for the bracket (2.1) to be super antisymmetric and satisfy the super Jacobi identity,

$$\begin{aligned} &\{F, G\}^{(\eta,\Lambda)} = -(-1)^{P(F)P(G)} \{G, F\}^{(\eta,\Lambda)} , \\ &(-1)^{P(F)P(K)} \left\{ \{F, G\}^{(\eta,\Lambda)} , K \right\}^{(\eta,\Lambda)} + \text{cycl.}(F, G, K) = 0 . \end{aligned} \tag{2.3}$$

Besides, the property takes place

$$\{F, GK\}^{(\eta,\Lambda)} = \{F, G\}^{(\eta,\Lambda)} K + (-1)^{P(F)P(G)} G \{F, K\}^{(\eta,\Lambda)} . \tag{2.4}$$

It is easily to see that

$$\Lambda^{AB}(\eta) = \left\{ \eta^A, \eta^B \right\}^{(\eta,\Lambda)} . \tag{2.5}$$

In case if

$$\Lambda^{AB} = E^{AB} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} ,$$

the generalized Poisson bracket (2.1) coincides with the ordinary super Poisson bracket,

$$\{F,G\}^{(\eta,E)} = \frac{\partial_r F}{\partial \eta^A} E^{AB} \frac{\partial_l G}{\partial \eta^B} = \{F,G\} \ . \tag{2.6}$$

If $\eta' = \eta'(\eta)$ is a nonsingular change of variables, then the generalized Poisson bracket (2.1) acquires in the primed variables the following form

$$\{F,G\}^{(\eta,\Lambda)} = \{F',G'\}^{(\eta',\Lambda')} = \frac{\partial_r F'}{\partial \eta'^A} \Lambda'^{AB}(\eta') \frac{\partial_l G'}{\partial \eta'^B} \ , \tag{2.7}$$

where

$$\begin{aligned} &F'(\eta') = F(\eta) \ , \quad G'(\eta') = G(\eta) \ , \\ &\Lambda'^{AB}(\eta') = \frac{\partial_r \eta'^A}{\partial \eta^C} \Lambda^{CD}(\eta) \frac{\partial_l \eta'^B}{\partial \eta^D} = \left\{\eta'^A, \eta'^B\right\}^{(\eta,\Lambda)} \ . \end{aligned} \tag{2.8}$$

By analogy with the case of the ordinary Poisson bracket one can ask the question: which kind of transformations preserves the generalized Poisson bracket forminvariant, namely when a relation holds

$$\Lambda'^{AB}(\eta') = \Lambda^{AB}(\eta') \ . \tag{2.9}$$

We will call such kind of transformations generalized canonical ones. They are just canonical transformations in case when the generalized Poisson bracket coincides with the ordinary Poisson bracket.

Consider transformations of the form

$$\eta' = e^{\check{W}} \eta \ . \tag{2.10}$$

In (2.10) the operator $\check{W}$ is defined by its action on functions of η,

$$\check{W} F(\eta) = \{F, W\}^{(\eta,\Lambda)} \ , \tag{2.11}$$

where $W(\eta)$, $(P(W) = 0)$, is a generating function of the transformation. We are going to demonstrate that the transformations (2.10) are just the generalized canonical transformations, connected continuously with the identical transformation. To this end one has, first, to verify that the following property takes place

$$e^{\check{W}}F(\eta) = F(e^{\check{W}}\eta) = F(\eta') \ . \tag{2.12}$$

Indeed, one can see, using (2.4), that

$$e^{\check{W}}F(\eta)e^{-\check{W}} = \sum_{n=0}^{\infty}\frac{1}{n!}[\check{W},[\check{W},\ldots,[\check{W},F]\ldots]] = e^{\check{W}}F(\eta) \ . \tag{2.13}$$

Then, one can write, for example, for any analytic function $F(\eta)$,

$$e^{\check{W}}F(\eta) = e^{\check{W}}F(\eta)e^{-\check{W}} = F\left(e^{\check{W}}\eta e^{-\check{W}}\right) = F\left(e^{\check{W}}\eta\right) = F(\eta') \ .$$

Now, let us introduce a function $F^{AB}(\alpha,\eta)$, $P(\alpha)=0$,

$$F^{AB}(\alpha,\eta) = \left\{e^{\alpha\check{W}}\eta^A, e^{\alpha\check{W}}\eta^B\right\}^{(\eta,\Lambda)} \ . \tag{2.14}$$

At $\alpha=0$ this function coincides with $\Lambda^{AB}(\eta)$, see (2.5), and at $\alpha=1$ with $\Lambda'^{AB}(\eta')$, see (2.8) and (2.10),

$$F^{AB}(0,\eta) = \Lambda^{AB}(\eta) \ , \tag{2.15}$$

$$F^{AB}(1,\eta) = \Lambda'^{AB}(\eta') \ . \tag{2.16}$$

Differentiating (2.14) with respect to α and using the Jacoby identity (2.3), one can get an equation for the function $F^{AB}(\alpha,\eta)$,

$$\frac{\partial F^{AB}(\alpha,\eta)}{\partial\alpha} = \check{W}F^{AB}(\alpha,\eta) \ . \tag{2.17}$$

A solution of this equation, which obeys the initial condition (2.15), has the form

$$F^{AB}(\alpha,\eta) = e^{\alpha\check{W}}\Lambda^{AB}(\eta) \ . \tag{2.18}$$

Taking into account the equation (2.16) and the property (2.12), we get just the condition (2.9) of the forminvariance of the generalized Poisson bracket. Thus, the transformations (2.10) are namely generalized canonical transformations, connected continuously with the identical transformation. By definition they preserve the forminvariance of the generalized Poisson bracket,

$$\{F,G\}^{(\eta,\Lambda)} = \{F',G'\}^{(\eta',\Lambda)}\ , \quad F'(\eta') = F(\eta),\ G'(\eta') = G(\eta)\ . \tag{2.19}$$

In particular, the infinitesimal form of the generalized canonical transformations is

$$\eta' = \eta + \delta\eta\ , \quad \delta\eta = \{\eta, \delta W\}^{(\eta,\Lambda)}\ . \tag{2.20}$$

Let us suppose now that the classical mechanics in question has equations of motion of the form

$$\dot{\eta} = \{\eta, H\}^{(\eta,\Lambda)}\ , \tag{2.21}$$

i.e. the hamiltonian equations of motion, but with a generalized Poisson bracket. How they are transformed under the generalized canonical transformations (2.20) ? The result is

$$\dot{\eta}' = \{\eta', H'\}^{(\eta',\Lambda)}\ , \quad H'(\eta') = H(\eta) + \frac{\partial \delta W}{\partial t}\ . \tag{2.22}$$

It means that the equations (2.21) are forminvariant under the generalized canonical transformations, only Hamiltonian is changed, similar to the usual case of the canonical transformations and hamiltonian equations of motion with the ordinary Poisson bracket. To see this, one has to calculate the time derivative of η', using (2.21),

$$\dot{\eta}' = \{\eta + \delta\eta, H\}^{(\eta,\Lambda)} + \left\{\eta, \frac{\partial \delta W}{\partial t}\right\}^{(\eta,\Lambda)} = \left\{\eta + \delta\eta, H + \frac{\partial \delta W}{\partial t}\right\}^{(\eta,\Lambda)}\ .$$

Taking into account (2.21),(2.20), and (2.19), we obtain just equations (2.22).

If a physical quantity is represented by a function $F(\eta)$ in the variables η then in the primed variables (2.10) it will be represented by a function $F'(\eta')$, which is related to the former one by the eq. $F'(\eta') = F(\eta)$. In the infinitesimal form it results in $F'(\eta) = F_{\delta W}(\eta)$, according the eq.(2.22),

$$F_{\delta W}(\eta) = F(\eta) + \delta F(\eta)\ , \quad \delta F(\eta) = \{\delta W, F\}^{(\eta,\Lambda)}\ . \tag{2.23}$$

Variations of the phase variables in course of the time evolution (2.20) can also be considered as a generalized canonical transformation. Namely, let η are the phase variables at a time instant t and η_0 are ones at the time instant $t = 0$. Then η are some functions

of η_0 and of t as a parameter, $\eta = \varphi(\eta_0, t)$. One can see that the transformation from η_0 to η is a generalized canonical transformation. Moreover, this transformation can be formally written explicitly. Indeed, considering for simplicity time independent Hamiltonians only, one can see that the solution of the Cauchy problem for the equation (2.20), with the initial data η_0 at $t = 0$, has the form

$$\eta = e^{\check{H}t}\eta_0 \ , \tag{2.24}$$

where the operator $\check{H}$ is defined by its action on functions $F(\eta_0)$ of η_0 as $\check{H}F(\eta_0) = \{F(\eta_0), H(\eta_0)\}^{(\eta_0,\Lambda)}$. Because of the transformation (2.24) is the generalized canonical transformation (see (2.10)) with the generating function $H(\eta_0)$, one has only prove that (2.24) obeys the equation of motion (2.20). Taking the time derivative from (2.24), one gets

$$\dot{\eta} = \check{H}e^{\check{H}t}\eta_0 = \left\{e^{\check{H}t}\eta_0, H(\eta_0)\right\}^{(\eta_0,\Lambda)} \ . \tag{2.25}$$

Using (2.12), one can verify that

$$H(e^{\check{H}t}\eta_0) = e^{\check{H}t}H(\eta_0) = H(\eta_0) \ . \tag{2.26}$$

Substituting (2.26) into (2.25) and taking into account the property (2.19), one obtains

$$\dot{\eta} = \left\{e^{\check{H}t}\eta_0, H(e^{\check{H}t}\eta_0)\right\}^{(\eta_0,\Lambda)} = \{\eta, H(\eta)\}^{(\eta,\Lambda)} \ ,$$

what proves our affirmation.

III. D-TRANSFORMATIONS

Now we are going to apply the previous consideration to theories with constraints, namely, with second-class constraints.

Let us consider a theory with second-class constraints $\Phi = (\Phi_l(\eta))$, in hamiltonian formulation, described by phase variables η^A, $A = 1, \ldots, 2n$, half of which are coordinates q and half are moments p, so that $\eta^A = (q^a, p_a)$, $A = (\zeta, a)$, $\zeta = 1, 2$, $a = 1, \ldots, n$. An important object in such theories is the Dirac bracket between two functions $F(\eta)$ and $G(\eta)$,

$$\{F, G\}_{D(\Phi)} = \{F, G\} - \{F, \Phi_l\}\{\Phi, \Phi\}^{-1}_{ll'}\{\Phi_{l'}, G\} \ . \tag{3.1}$$

It is easy to see that the Dirac bracket is a particular case of the generalized Poisson bracket (2.1),

$$\{F, G\}_{D(\Phi)} = \{F, G\}^{(\eta,\Lambda)} \ , \tag{3.2}$$

with

$$\Lambda^{AB} = E^{AB} - \{\eta^A, \Phi_l\}\{\Phi, \Phi\}^{-1}_{ll'}\{\Phi_{l'}, \eta^B\} = \{\eta^A, \eta^B\}_{D(\Phi)} \ . \tag{3.3}$$

If so, then one can consider the generalized canonical transformations for such a generalized Poisson bracket. This special but important case of the generalized canonical transformations we will call D-transformations. Thus, by the definition, D-transformations $\eta \to \eta'$ preserve the forminvariance of the Dirac bracket[1],

$$\{F, G\}_{D(\Phi)} = \{F', G'\}'_{D(\Phi)} \ . \tag{3.4}$$

As we will see further, in theories with second-class constraints, D-transformations play the same role which play canonical transformations in theories without constraints.

An explicit form of D-transformations connected continuously with the identical transformation can be extracted from (2.10) and (3.2),

$$\eta' = e^{\check{W}}\eta \ , \quad \check{W}F(\eta) = \{F, W\}_{D(\Phi)} \ , \tag{3.5}$$

and in the infinitesimal form

$$\eta' = \eta + \delta\eta \ , \quad \delta\eta = \{\eta, \delta W\}_{D(\Phi)} \ , \tag{3.6}$$

where $W(\eta)$ is a generating function of the D-transformation.

[1] A prime above the Dirac bracket in (3.4) means that the latter is calculated in the primed variables.

One can see that D-transformations differ from canonical ones only by terms proportional to constraints. Indeed, the variation $\delta\eta$ under the D-transformation can be written as

$$\delta\eta = \{\eta, \delta W\}_{D(\Phi)} = \{\eta, \delta W'\} + \{\Phi\} \ , \tag{3.7}$$

where

$$\delta W' = \delta W - \Phi_l \{\Phi, \Phi\}^{-1}_{ll'} \{\Phi_{l'}, \delta W\} \ ,$$

and $\{\Phi\}$ accumulates terms proportional to constraints, or terms which vanish on the constraint surface.

As is known [8] equations of motion for a theory with second-class constraints can be written in the form

$$\dot{\eta} = \{\eta, H\}_{D(\Phi)} \ , \tag{3.8}$$

$$\Phi(\eta) = 0 \ . \tag{3.9}$$

They consist of two groups of equations, hamiltonian equations (3.8) with the Dirac bracket, which is in the same time a generalized Poisson bracket, and equations of constraints (3.9). Using the previous section consideration, one can say that the equations (3.8) are forminvariant under the D-transformations. It turns also out that the equations of constraints (3.9) are forminvariant under the D-transformations. Indeed, let $\Phi'(\eta') = 0$ are equations of constraints in variables η', connected with η by a D-transformation, then the relations

$$\Phi'(\eta') = \Phi(\eta) \tag{3.10}$$

have to hold. One can consider these relations as functional equations for the functions Φ'. It is easily to verify that they have a solution $\Phi' = \Phi$. Indeed, consider the functions $\Phi(\eta')$. Using the formula (2.12) and a well known property of the Dirac bracket: $\{F, \Phi_l\}_{D(\Phi)} = 0$ for any function $F(\eta)$ and any constraint Φ_l, we get

$$\Phi(\eta') = e^{\check{W}} \Phi(\eta) = \Phi(\eta) \ . \tag{3.11}$$

That means that the constraints surface $\Phi(\eta) = 0$ after the D-transformation can be described by the same functions, i.e. by the equations $\Phi(\eta') = 0$.

Thus, equations of motion of theories with second-class constraints are forminvariant under the D-transformations. Namely, the equations (3.8) and (3.9) have the following form after the D-transformation (3.6):

$$\dot{\eta}' = \{\eta', H'\}'_{D(\Phi)} \;, \quad \Phi(\eta') = 0 \;, \quad H'(\eta') = H(\eta) + \frac{\partial \delta W}{\partial t} \;, \tag{3.12}$$

or

$$\dot{\eta} = \left\{\eta, H_{\delta W} + \frac{\partial \delta W}{\partial t}\right\}_{D(\Phi)} \;, \quad \Phi(\eta) = 0 \;, \tag{3.13}$$

and the physical quantities F are described by the functions $F_{\delta W}(\eta)$, see (2.23),

$$F'(\eta) = F_{\delta W}(\eta) = F(\eta) + \{\delta W, F\}_{D(\Phi)} \;, \tag{3.14}$$

In the special canonical variables (ω, Ω), in which equations of constraints have a simple form $\Omega = 0$ (see [9,10]), and the Dirac bracket reduces to the Poisson one in the variables ω, so that the latter are physical variables on the constraints surface, D-transformations have a simple meaning: they are canonical transformations in the sector of physical variables ω with no change of variables Ω. It is natural because the D-transformations do not change the form of constraints.

IV. QUANTUM IMPLEMENTATION OF D-TRANSFORMATIONS

One can ask a question: which kind of transformations in quantum theory corresponds to D-transformations in classical theory? It is easy to see that these are unitary transformations and vice versa: unitary transformations in a quantum theory with constraints induce in a sense D-transformations in the corresponding classical theory. From this point of view D-transformations in theories with constraints play also the role similar to one of the canonical transformations in theories without constraints. To prove this affirmation we have to remember that in a classical theory D-transformations are transformations of trajectories-states of

the theory. Thus, if to speak literally, some transformations of quantum states-vectors in a Hilbert space, have to correspond them in a quantum theory.

Let us have a classical theory with second-class constraints, which is described by the equations of motion (3.8,3.9). Its canonical quantization [8,9] consists formally in a transition from the classical variables η to quantum operators $\hat{\eta}$, $P(\hat{\eta}^A) = P(\eta^A) = P_A$, which obey the operator relations[2]

$$[\hat{\eta}^A, \hat{\eta}^B\} = i\hbar \widehat{\overline{\{\eta^A, \eta^B\}_{D(\Phi)}}} = i\hbar \widehat{\overline{\Lambda^{AB}(\eta)}}, \quad \widehat{\overline{\Phi(\eta)}} = 0, \tag{4.1}$$

and which suppose to be realized in a Hilbert space $\mathcal{R}$ of vectors $|\Psi>$. Then one has to assign operators $\hat{F}$ to all the physical quantities F, which are described in the classical theory by the functions $F(\eta)$, using a certain correspondence rule, $\hat{F} = \widehat{\overline{F(\eta)}}$. The time evolution of the state vectors is defined by the quantum Hamiltonian $\hat{H} = \widehat{\overline{H(\eta)}}$, according the Schrödinger equation

$$i\hbar \frac{\partial |\Psi>}{\partial t} = \hat{H}|\Psi> \; . \tag{4.2}$$

Let us consider a unitary transformation of the state vectors, $|\Psi> \rightarrow |\Psi'> = \hat{U}|\Psi>$, where $\hat{U}$ is some unitary operator, $\hat{U}^+\hat{U} = 1$, which one can write in the form

$$\hat{U} = \exp\left\{-\frac{i}{\hbar}\hat{W}\right\} \; , \tag{4.3}$$

where $\hat{W}$ is a hermitian operator, $\hat{W}^+ = \hat{W}$, further called quantum generator of the transformation. In the infinitesimal form ($\hat{W} \rightarrow \delta\hat{W}$), simplifying the consideration, $|\Psi'> = |\Psi> + \delta|\Psi>$, $\delta|\Psi> = -\frac{i}{\hbar}\delta\hat{W}|\Psi>$.

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[2]Via $[\hat{A}, \hat{B}\}$ we denote a generalized commutator of two operators $\hat{A}$ and $\hat{B}$, with definite parities $P(\hat{A})$ and $P(\hat{B})$, $[\hat{A}, \hat{B}\} = \hat{A}\hat{B} - (-1)^{P(\hat{A})P(\hat{B})}\hat{B}\hat{A}$. An overline with a hat, above a classical function $A(\eta)$, here and further means a certain rule of correspondence between the function and the corresponding quantum operator $\hat{A}$, $\hat{A} = \widehat{\overline{A(\eta)}}$. The former is in this case the symbol of the operator [11]. A choice of this rule is not important in our considerations.

One can find a variation of operators of physical quantities from the condition $<\Psi|\hat{F}|\Psi>=<\Psi'|\hat{F}'|\Psi'>$, which results in

$$\hat{F}' = \hat{F}_{\delta W} = \hat{U}\hat{F}\hat{U}^{+} = \hat{F} + \delta\hat{F}, \quad \delta\hat{F} = -\frac{i}{\hbar}[\delta\hat{W}, \hat{F}] \ . \tag{4.4}$$

If $\delta W(\eta)$ is a symbol of the operator $\delta\hat{W}$, $\delta\hat{W} = \widehat{\overline{\delta W \eta)}}$ and $F(\eta)$ is one of the operator $\hat{F}$ (the classical function which describes the physical quantity in the variables η), $\hat{F} = \widehat{\overline{F(\eta)}}$, then it follows from the eq. (4.1)

$$\delta\hat{F} = \widehat{\overline{\{\delta W, F\}}}_{D(\Phi)} + o(\hbar) \ . \tag{4.5}$$

Remembering the formula (3.14), one can write

$$\hat{F}_{\delta W} = \widehat{\overline{F_{\delta W}(\eta)}} + o(\hbar) \ . \tag{4.6}$$

Thus, operators of physical quantities, transformed in course of a unitary transformation, have as their symbols initial classical functions transformed by a D-transformation, with the generating function being a classical symbol of the quantum generator of the unitary transformation.

The Schrödinger equation for transformed vectors can be derived from the eq. (4.2) and has the form

$$i\hbar\frac{\partial|\Psi'>}{\partial t} = \hat{H}'|\Psi'> \ , \quad \hat{H}' = \hat{H}_{\delta W} + \frac{\partial}{\partial t}\widehat{\overline{\delta W}} \ . \tag{4.7}$$

Thus, the time evolution of the state vectors after the unitary transformation is governed by a quantum Hamiltonian with the classical symbol

$$H'(\eta) = H_{\delta W}(\eta) + \frac{\partial \delta W(\eta)}{\partial t} + o(\hbar) \ . \tag{4.8}$$

That fact and the eq.(4.1) allow one to see that the classical limit of the quantum theory after the unitary transformation (4.3) is described by the equations (3.13) and therefore corresponds to the D-transformed classical theory with the generating function, which is a classical symbol of the quantum generator of the unitary transformation. In the same way

one can prove an inverse statement: if we have a classical theory and its D-transformed formulation, then quantum versions of both theories are connected by an unitary transformation. Besides, the classical generating function of the D-transformation and the quantum generator of the unitary transformation are connected in the above mentioned menner.

Consider now the generating functional $Z(J)$ of Green's functions for theory with second-class constraints in the form of hamiltonian path integral and a behavior of the latter under the D-transformations. Such an integral can be written in the form

$$Z(J) = \int \exp\left\{\frac{i}{\hbar}S_J(\eta)\right\}\mathcal{D}\eta\ , \tag{4.9}$$

where

$$S_J(\eta) = \int [p_a\dot{q}^a - H_J(\eta)]dt, \quad H_J(\eta) = H(\eta) + J_A\eta^A\ ,$$

is the classical action with sources, $J_A(t)$ are sources to the variables $\eta^A(t)$, $\quad P(J_A) = P(\eta^A) = P_A$, and the measure $\mathcal{D}\eta$ has the form [12,13],

$$\mathcal{D}\eta = \mathrm{Sdet}^{1/2}\{\Phi,\Phi\}\delta(\Phi)D\eta\ , \tag{4.10}$$

with $\mathrm{Sdet}\{\Phi,\Phi\}$ denoting the superdeterminant of the matrix $\{\Phi_l,\Phi_m\}$.

As is known, if a change of variables $\eta' = \eta'(\eta)$ is a canonical transformation, then $|\mathrm{Ber}\ \eta'(\eta)| = 1$, where $\mathrm{Ber}\eta'(\eta)$ is Berezinian [11] of the change of variables, $\mathrm{Ber}\ \eta'(\eta) = \mathrm{Sdet}\partial_r\eta'^A/\partial\eta^B$. In particular, for infinitesimal canonical transformations $\eta' = \eta + \delta\eta,\ \ \delta\eta = \{\eta,\delta W\},\quad \mathrm{Ber}\ \eta'(\eta) = 1$. In case of theories without constraints, the measure $\mathcal{D}\eta$ (4.10) reduces to $D\eta$ and is invariant under canonical transformations, but in theories with constraints it is not. However, this measure is invariant under D-transformations,

$$\mathcal{D}\eta' = \mathcal{D}\eta\ ,$$

which confirms ones again that the latter play the role of canonical transformations in theories with constraints. To see this one can use a relation [10],

$$\mathrm{Sdet}^{1/2}\{\Phi,\Phi\}\delta(\Phi)\Big|_{\eta\to\eta'(\eta)}\mathrm{Ber}\ \eta'(\eta) = \mathrm{Sdet}^{1/2}\{\Phi,\Phi\}\delta(\Phi)\ , \tag{4.11}$$

where $\eta' = \eta + \{\eta, \delta W\}_{D(\Phi)}$.

The invariance of the measure (4.10) under D-transformations, induces an invariance of the integral (4.9) under the transformations of the action $S_J(\eta)$,

$$S_J(\eta) \to S'_J(\eta) = S_J(\eta'(\eta)) = S_J(\eta) + \delta S_J(\eta) \ , \tag{4.12}$$

where $\eta'(\eta)$ are D-transformations,

$$Z(J) = \int \exp\left\{\frac{i}{\hbar} S_J(\eta)\right\} \mathcal{D}\eta = \int \exp\{\frac{i}{\hbar} S'_J(\eta)\}\mathcal{D}\eta \ ,$$

or

$$\int \delta S_J(\eta) \exp\left\{\frac{i}{\hbar} S_J(\eta)\right\} \mathcal{D}\eta = 0 \ . \tag{4.13}$$

It is enough to know $\delta S_J(\eta)$ on the constraints surface, because of the integration in (4.13) is only going over this surface due to the δ-function in the measure (4.10). Taking into account the representation (3.7), one can find an expression for $\delta S_J(\eta)$ on the constraints surface,

$$\delta S_J(\eta)|_{\Phi=0} = (p\delta q - \delta W)|_{t_{in}}^{t_{out}} + \int \left[\frac{\partial}{\partial t}\delta W - \{H_J, \delta W\}_{D(\Phi)}\right] dt \ . \tag{4.14}$$

In field theory usually $t_{in,out} \to \pm\infty$ and trajectories of integration vanish at these time limits. Considering D-transformations, which do not change this property, one gets

$$\int \left[\int \left(\frac{\partial}{\partial t}\delta W - \{H_J, \delta W\}_{D(\Phi)}\right) dt\right] \exp\left\{\frac{i}{\hbar} S_J(\eta)\right\} \mathcal{D}\eta = 0 \ . \tag{4.15}$$

This relation can be used to obtain different kinds of equations for generating functional and therefore for Green's functions. For example, let us consider D-transformations with two types of generating functions: $\delta W = \epsilon_A \eta^A$, and $\delta W = \zeta_l \Phi_l(\eta)$, with arbitrary "small" time dependent functions $\epsilon_A(t)$ and $\zeta_l(t)$. Using these δW in eq. (4.15), we get two relations

$$\begin{aligned} &\int \left[\dot{\eta}^A - \{\eta^A, H_J\}_{D(\Phi)}\right] \exp\left\{\frac{i}{\hbar} S_J(\eta)\right\} \mathcal{D}\eta = 0 \ , \\ &\int \Phi_l(\eta) \exp\left\{\frac{i}{\hbar} S_J(\eta)\right\} \mathcal{D}\eta = 0 \ , \end{aligned} \tag{4.16}$$

which can be rewritten in the form of Schwinger equations for the functional $Z(J)$,

$$\left[\dot{\eta}^A - \{\eta^A, H_J\}_{D(\Phi)}\right]_{\eta \to \frac{\delta_l}{\delta(iJ)}} Z(J) = 0 \ , \quad \Phi\left(\frac{\delta_l}{\delta(iJ)}\right) Z(J) = 0 \ . \tag{4.17}$$

V. REMARKS

Thus, we demonstrated that in theories with second-class constraints D-transformations play the usual role of canonical ones. In fact, in our books [9] we have already used infinitesimal D-transformations for technical reasons, but at that time we did not fully realize their special role.

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